III. *On the necessary Truth of certain Conclusions obtained by Means of imaginary Quantities.* By Robert Woodhouse, *A. M. Fellow of Caius College. Communicated by the Rev.* S. Vince, *A. M. Plumian Professor of Astronomy in the University of Cambridge.*

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Amongst the various objections urged against mathematical science, few oppose its evidence and logical accuracy; and, since its demonstrations have been acknowledged to proceed by a series of the strictest inferences, from the most evident principles, the study of abstract science has generally been deemed peculiarly proper to habituate the mind to just reasoning. But of late, the dissentions of mathematicians have subjected to doubt, even this "collateral and intervenient use;" for, not only has the mode of applying analysis to physical objects been controverted, but certain parts of the pure mathematics have become the subject of dispute. Much has been heard of the science of quantity being vitiated with jargon, absurdity, and mystery, and perplexed with paradox and contradiction; so that, from the very complaints of the patrons of mathematics, its opponents may derive their most potent arguments, and abundant matter for triumphant invective.

The introduction of impossible quantities, is assigned as a great and primary cause of the evils under which mathematical science labours. During the operation of these quantities,

it is said, all just reasoning is suspended, and the mind is bewildered by exhibitions that resemble the juggling tricks of mechanical dexterity.

The arguments that seem to render all operations performed with impossible quantities unintelligible, may be included under the following statement. Algebra is a species of short-hand writing; a language, or system of characters or signs, invented for the purpose of facilitating the comparison and combination of ideas. Now all demonstration by signs, must ultimately rest on observations made on individual objects; and all the varieties of the transformation and combination of signs, except what are arbitrary and conventional, must be regulated by properties observed to belong to the things of which the signs are the representatives. Demonstration by signs is shewn to be true, by referring to the individual things the signs represent; and is shewn to be general, by remarking that the operation is the same, whatever is the thing signified, or, in other words, that the operation is independent of the things signified. Yet, against this statement, from the very concessions of the mathematicians that have opposed the use of impossible quantities, is to be derived a powerful argument, an argument sufficiently satisfactory to the mind, that operations with impossible quantities are really regulated by the rules of a logic equally just with the logic of possible quantities. It is conceded, and mentioned as a paradox, that the conclusions obtained by the aid of imaginary quantities are most true and certain. Now, if operations with any characters or signs lead to just conclusions, such operations must be true by virtue of some principle or other; and the objections against imaginary quantities, ought to be diverted upon the unsatisfactory explanation given of their

nature and uses. It would indeed be a singular paradox, or a rare felicity, if truth, not always attained by meditation, should unexpectedly result from un-ideal operations conducted without principle, purpose, or regularity.

The paradox, that a process in which no idea is introduced conducts to truth, and that operations by unintelligible characters lead to certain and just conclusions, has been expressly treated in a paper presented to the Royal Society. The ingenious author, confining his enquiry concerning impossible quantities to their use in calculating the values of sines, cosines, &c. has attempted to shew, that operations with such quantities are true, on the principle of analogy. He is of opinion, that, "The operations performed with imaginary charac- tacters, though destitute of meaning themselves, are yet notes of reference to others which are significant. They point out indirectly a method of demonstrating a certain property of the hyperbola, and then leave us to conclude from analogy, that the same property belongs also to the circle. All that we are assured of by the imaginary investigation is, that its conclusion may, with all the strictness of mathematical reasoning, be proved of the hyperbola; but if from thence we would transfer that conclusion to the circle, it must be in consequence of the principle just now mentioned. The investigation therefore resolves itself ultimately into an argument from analogy; and, after the strictest examination, will be found without any other claim to the evidence of demonstration." By this explanation, the operations of imaginary quantities, before disorderly and confused, assume some appearance of purpose and regularity; and the assent of the mind, if not compelled by certain proof, is at least solicited by probable arguments. But, to mathe-

maticians, who, in questions of abstract science, profess never to rest contented with "a rational faith and moral persuasion," the principle of explanation just adduced must needs be unsatisfactory; for, whatever extension of meaning be allowed to the term analogy, still this is certain, that a proof by analogy is inferior to strict demonstration. What is it that determines the nature of this analogy? Or how can its several coincidences, interruptions, and limitations be ascertained, except by separate and direct investigations of the properties of the circle and hyperbola? If the analogy between the two curves depends on investigation, and is limited thereby, then all operations with imaginary expressions are perfectly nugatory; since we are not warranted to adopt a single conclusion obtained by their aid, except such conclusion be verified by a distinct and rigorous demonstration.

The author of the principle of analogy allows that it is imperfect; and I perceive no sure method of ascertaining the restrictions to which it is subject, except by the forms that result from actual investigation.

To shew that the principle of analogy ought to be abandoned, and a more natural and satisfactory one sought for, an argument may be used, similar to the one employed against those who maintain operations by imaginary symbols to be perfectly unintelligible; that, since arguments have been invented, which, if they do not satisfy, yet afford the mind a glimpse and indistinct perception of the reason why certain processes lead to truth, it may be presumed possible to convert such probable arguments into certain proofs, and to discipline a vague, perilous, and irregular analogy, into a strict, sure, and formal demonstration.

Convinced in my own mind, that there can be neither paradoxes nor mysteries inherent and inexplicable in a system of characters of our own invention, and combined according to rules, the origin and extent of which we can precisely ascertain, I have endeavoured, in the present memoir, to shew why certain conclusions obtained through the means of imaginary quantities are necessarily true: to effect this is my prime object; a subordinate one is, to shew that the method founded on imaginary symbols is commodious, and proper to be adopted, because of easy and extensive application.

It has been already observed, that demonstration ultimately depends on observations made on individual objects, and that a conclusion expressed by certain characters and signs, if general, must be true in each particular case that presents itself, on assigning specific values to the signs. After affixing a signification to the symbols $\times$, $+$, &c. the product of $(a+b)$ and $(c+d)$ can be proved equal to $(ac)+(ad)+(bc)+(bd)$; if $na=b$, a can be proved equal $\frac{b}{n}$, a, b, c, &c. being the signs of real quantities; but nothing can be affirmed concerning the product of $(a+b\sqrt{-1})$, and $(c+d\sqrt{-1})$, nor concerning the form $na=b\sqrt{-1}$; and all that can be meant by the form $(a+b\sqrt{-1})\times(c+d\sqrt{-1})$ is, that the characters are to be combined after the same manner that the signs of real quantities are; so that $(a+b\sqrt{-1})\times(c+d\sqrt{-1})$, and $ac+ad\sqrt{-1}+cb\sqrt{-1}-bd$, are two forms equivalent to each other, not proved equivalent, but put so, by extending the rule demonstrated for the signs of real quantities to characters that are insignificant.

In like manner $(a+b)^{x\sqrt{-1}}$ can never be proved equal to $a^{x\sqrt{-1}} + x\sqrt{-1}\,a^{x\sqrt{-1}-1}\,b + \&c.$ it is only an abridged symbol for the series; there can be no ambiguity in the meaning of $(a+b)^{x\sqrt{-1}}$, since it is intended to represent the series which arises from developing $(a+b)^{x\sqrt{-1}}$, after the same manner that $(a+b)^{x}$ is developed.

The symbol $\sqrt{-1}$ might arise from translating questions of which the statement involved a contradiction of ideas into algebraic language, and reasoning on them, as if they really admitted a solution. For instance, if it were required to divide the number 12 into two such parts, that their product should equal 37, this question in algebraic language would be $12x - x^2 = 37$; an absurd statement, since no real number can be assigned to x that verifies it; but, according to the rules for transposition, the equation $12x - x^2 = 37$, is equivalent to $x^2 - 12x + 36 = -1$. If x were the sign of a real quantity, $x-6$, or $6-x$, would be the square root of $x^2 - 12x + 36$; if therefore $\pm(x-6)$ be put for the square root, it is put so by extending the rule proved for real quantities to this case; and the radical placed over the symbol -1, shews that such extension has been assumed; hence $x - 6 = +\sqrt{-1}$ is an expression of which the origin is known, being derived from $x^2 - 12x + 36 = -1$.

In the present inquiry, it is immaterial how the symbol $\sqrt{-1}$ originated: I think its origin most probably accounted for thus. The determination of general rules for the combination of algebraic quantities, was probably posterior to the actual solution of many problems, effected by particular artifices. During the solutions, certain similar parcels of characters presented themselves,

which it was necessary either to combine or separate; and, to obtain general rules for their combination and separation, the first algebraists feigned forms similar to what really presented themselves in specific cases:* thus, in questions producing

* It has been already observed, that the determination of general rules for algebraic operations was posterior to the actual solutions of problems. To obtain a rule for the multiplication of algebraic quantities, a form such as $a - b + c - m$, was proposed to be multiplied by $d - e + f - n$; since it was necessary to have a law for the multiplication of the signs $+ \; -$, a general one was established, that like signs multiplied produce $+$, unlike $-$, either from proving such law when $(a + c)$ was $> b + m$, and $(d + f) > (e + n)$, or from remarking that, in the solution of problems, the observance of such law always produced true conclusions. It is very certain that the mind can form no idea of an abstract negative quantity; and therefore nothing can be affirmed concerning the multiplication of $-a$, and $-b$, nor of $,a - b$, and $,c - d$, if a is $< b$, or $e < d$. Let us attend, however, to the real meaning of negative quantities, and to the cause of their appearance in the solution of problems. The rule for transposition is, that quantities may be transferred from one side of the equation to the other, changing their signs. By virtue of this rule, an equation may appear under the form $-x = a - b$, $(a < b)$, or $x - y = a - b$, $x < y$, $a < b)$; which equations, abstractedly considered, may appear absurd, but become intelligible by means of the equations $x = b - a$, $y - x = b - a$, to which they are significant, and to which they may be immediately reduced. Suppose now, $-x = a - b$ is to be multiplied by $-z = m - n$ $(m < n)$; if the forms be reduced to their equivalent ones, $x = b - a$, and $z = n - m$, and then multiplied, the product may be proved $xz = bn - bm - an + am$. Now, let $(a - b)$ be multiplied by $(m - n)$, in the same manner as it ought to be if a were $> b$, $m > n$, and the product is, $(am - an - bm + bn)$, or $(bn - bm - an + am)$, the same as arose from multiplying $b - a$ by $n - m$, and which is equal xz; hence $-x \times -z$ must be put xz; hence, in multiplication, we are sure to have right results by always observing the law that the product of like signs is $+$, of unlike $-$. In a similar manner it may be proved, that the product of $x - y = a - b$, by $m - n = d - c$, will be truly expressed by combining the quantities according to the same law for the signs. It is evident how much the establishment of this law must facilitate calculation; since, without considering whether x is greater or less than y, the product of $(x - y)$ and of $(m - n)$ may immediately be put down. Such equations as $x - y = a - b$, $(< y)$ must frequently occur in calculation, unless every step of the process be rendered extremely

quadratic equations, forms such as $x^2 - 7x + 10$, $x^2 + 3x - 10$, appeared; and therefore, to obtain a general rule for the solution of all like forms, $x^2 \pm ax \pm b$ was invented; and the solution, being made general, was necessarily extended to those cases which admitted no real answer. When such an extension is assumed, it is always indicated by the symbol $\sqrt{-1}$; and hence, to know what operations are to be performed with the symbol $\sqrt{-1}$, it is necessary to recur to the quadratic forms from which it is arbitrarily derived.

I now proceed to shew how sines, cosines, &c. may be expressed by means of exponential expressions; and, for the sake of perspicuity, I avoid all fluxionary operations, and adhere to a purely algebraical calculus.

To find the form for the developement of e^x, let $y = e^x$,

or $y = \overline{1 + \overline{e - 1}}|^{x} = \overline{\overline{1 + \overline{e - 1}}|^{n}}|^{\frac{x}{n}}$, n being any quantity which disappears of itself in the value of y.

Now $\overline{1 + \overline{e - 1}}|^{n} = 1 + n(e - 1) + \frac{n \cdot (n - 1)}{2}(e - 1)^2 +$ &c. $=$ (arranging the terms according to the powers of n) $1 + An + Bn^2 + Cn^3 +$, &c.

$A = (e - 1) - \frac{1}{2}(e - 1)^2 + \frac{1}{3}(e - 1)^3 -$ &c. the values of B, C, &c. it is unnecessary to investigate, since they disappear in the calculation.

tedious by considerations on the relative value of quantities, and unless the rule for transposition be clogged with needless limitations: an abstract negative quantity is indeed unintelligible; but $-x = -a$, or $x - y = a - b$ $(x < y)$, are perfectly intelligible by means of their equivalent equations, $x = a$, $y - x = b - a$, to which they can be immediately reduced. The tendency of the reform proposed to be introduced into algebra is, it appears to me, to destroy the chief advantages of that art; its compendious and expeditious methods of calculation.

Hence $y = (1 + An + Bn^2 + Cn^3 + \&c.)^{\frac{x}{n}}$

$$= 1 + \frac{x}{n}(An + Bn^2 + \&c.) + \frac{x.(x-n)}{2n^2}(An + Bn^2 + \&c.)^2$$

$$= 1 + x(A + Bn + \&c.) + \frac{x.(x-n)}{2}(A \quad + Bn + \&c.)^2.$$

Now, since n is arbitrary, and ought, by the nature of the function y, to disappear from the expression of the function, it follows, that all terms multiplied by each power of n must destroy each other; neglecting, therefore, the terms which ought of themselves to disappear, whatever n is, we have simply,

$$y = e^x = 1 + Ax + \frac{A^2 x^2}{1.2} + \frac{A^3 x^3}{1.2.3} + \&c.$$

$$\text{if } A = 1) = 1 + \quad x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.*$$

This demonstration for the developement of e^x is general, whatever x is, provided it is always the sign of a real quantity; but $e^{x\sqrt{-1}}$ can never be proved equal to $1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \&c.$† What then is to be understood by $e^{x\sqrt{-1}}$? merely this, that $e^{x\sqrt{-1}}$ is an abridged symbol for the series of characters $1 + x\sqrt{-1} - \frac{x^2}{1.2} - \&c.$ not proved, but assumed, by extending the form really belonging to e^x to $e^{x\sqrt{-1}}$.

In like manner, $e^{-x\sqrt{-1}}$ is an abridged symbol for

* This demonstration is due to M. LAGRANGE.

† In all treatises, after the demonstration for the developement of e^x, $e^{x\sqrt{-1}}$ is put $1 = x\sqrt{-1} - \frac{x^2}{1.2} - \&c.$ as if this case was really included in the general one of e^x.

$1 - x\sqrt{-1} - \frac{x^2}{1.2} -$, &c. $e^{(x+y)\sqrt{-1}}$ an abridged symbol for $1 + (x+y)\sqrt{-1} - \frac{(x+y^2)}{1.2}$ — &c. and there can be no ambiguity in what these symbols are meant to represent; since we have only in the demonstrated form $1 + x + \frac{x^2}{1.2} +$ &c. to substitute $x\sqrt{-1}$, $-x\sqrt{-1}$, or $(x+y)\sqrt{-1}$, for x.

The use made of these abridged symbols is, to express, in an algebraic form, certain lines belonging to the circle, as sines, cosines, &c. for, since

$e^{x\sqrt{-1}}$ is an abridged symbol for $1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3}$, &c.

and $e^{-x\sqrt{-1}}$ for $1 - x\sqrt{-1} - \frac{x^2}{1.2} + \frac{x^3\sqrt{-1}}{1.2.3}$, &c.

$\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}$ is a symbol for $1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4}$ — &c.

$\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$ is a symbol for $x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5}$ — &c.

but $1 - \frac{x^2}{1.2} +$ &c. and $x - \frac{x^3}{1.2.3} +$ &c. represent the cosine and sine of an arc x $\therefore$ $\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}$, and * $\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$, in consequence of the assumptions made, properly represent the sine and cosine of an arc x.

* The usual method of deducing these expressions, is by a fluxionary process. I have preferred an algebraical one, for the sake of perspicuity. In an algebraical investigation, every step may be closely examined, and we can easily retrace to the original notions from which it commenced. In fluxions, the significancy of the expressions, and the nature and manner of their derivation, demand much time and attention, to be properly understood. If, however, the fluxionary process be examined, its object will appear to be, to find out a method of abridgedly representing the sine, &c. of an arc, employing for that purpose a form demonstrated for real quantities. In this fluxionary process, it is quite unnecessary to mention either the hyperbola or logarithms.

To remove all doubt and occasion of cavil, it is to be understood, that $(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}})$ means, that the terms of the series which $e^{x\sqrt{-1}}$ represents, are to be connected with the terms of the series that $e^{-x\sqrt{-1}}$ represents, according to the rules obtaining for the addition of real quantities: again, that $x\sqrt{-1} - x\sqrt{-1}$ is put equal 0, not by bringing $x\sqrt{-1}$ under the predicament of quantity, and making it the subject of arithmetical computation, but by giving to $+$ and $-$ their proper signification when used with real quantities, and then they designate reverse operations: again, that $\frac{x\sqrt{-1}}{\sqrt{-1}}$ is equal to x, not because it is true that a quantity multiplied and divided by the same number remains the same, but because $\frac{x\sqrt{-1}}{\sqrt{-1}}$ means, that x is to be combined with $\sqrt{-1}$ after the manner that real quantities are in multiplication, and then divided after the manner that real quantities are in division; and therefore, since the two operations are the reverse of each other, $\frac{x \times \sqrt{-1}}{\sqrt{-1}}$ and x must be equivalent expressions.*

To facilitate the solution of the propositions demonstrated by means of imaginary quantities, I previously observe, that, A being any symbol whatever, $A \times (e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} - e^{y\sqrt{-1}})$, and $A\,e^{x\sqrt{-1}} + A\,e^{-x\sqrt{-1}} - A\,e^{y\sqrt{-1}}$, are equivalent ex-

* After this manner ought to be interpreted what MACLAURIN and BERNOUILLI have indistinctly expressed, concerning the compensation that ought to take place when real quantities are represented by means of imaginary symbols.—BERN. Vol. I. No. 70. MACL. Fluxions, Art. 699, 763.

pressions; for the same series results, whether the terms of the developement for $e^{x\sqrt{-1}}$, $e^{-x\sqrt{-1}}$, $e^{y\sqrt{-1}}$ be connected together after the manner pointed out by the signs $+$ $-$, and then combined with A, or whether A be first separately combined with each term of $e^{x\sqrt{-1}}$, $e^{-x\sqrt{-1}}$, $e^{y\sqrt{-1}}$, and then the resulting terms added together: again, $e^{x\sqrt{-1}} \times e^{y\sqrt{-1}}$, and $e^{(x+y)\sqrt{-1}}$, are equivalent expressions; for the same series results, whether the terms for the developements of $e^{x\sqrt{-1}}$ and $e^{y\sqrt{-1}}$ be connected together after the manner of quantities in multiplication, or whether $e^{(x+y)\sqrt{-1}}$ be immediately developed, by putting $(x+y)\sqrt{-1}$ for x, in the series $1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$

for $e^{x\sqrt{-1}}$ is the symbol for $1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \&c.$

$e^{y\sqrt{-1}}$ is the symbol for $1 + y\sqrt{-1} - \frac{y^2}{1.2} - \frac{y^3\sqrt{-1}}{1.2.3} + \&c.$

$\therefore$ $e^{x\sqrt{-1}} \times e^{y\sqrt{-1}}$ (the symbol $\times$ indicating that the several terms are to be connected together according to the rules of multiplication) equals $1 + (x+y)\sqrt{-1} - \left(\frac{(x+y)^2}{2}\right) - \&c.$ which series is abridgedly expressed by the symbol $e^{(x+y)\sqrt{-1}}$.

$\therefore$ $e^{x\sqrt{-1}} \times e^{y\sqrt{-1}}$, and $e^{(x+y)\sqrt{-1}}$, are symbols alike significant; or, since it must now be evident in what sense the equality of imaginary expressions is to be understood, $e^{x\sqrt{-1}} \times e^{y\sqrt{-1}} = e^{(x+y)\sqrt{-1}}$.

After this explanation of the nature of the operations directed by means of certain signs $\times$, $+$, &c. to be performed with the symbols $e^{x\sqrt{-1}}$, $e^{y\sqrt{-1}}$, &c. the following propositions may be clearly and strictly proved.

1. Sin. x . cos. $y = \frac{1}{2}$ sin. $(x + y) + \frac{1}{2}$ sin. $(x - y)$.

For sin. $x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$, cos. $y = \frac{e^{y\sqrt{-1}} + e^{-y\sqrt{-1}}}{2}$;

$\therefore$ sin. $x \times$ cos. $y = \left(\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}\right) \times \left(\frac{e^{y\sqrt{-1}} + e^{-y\sqrt{-1}}}{2}\right)$

= by what has been shewn,

$$\tfrac{1}{2} \times \frac{e^{x\sqrt{-1}}}{2\sqrt{-1}} \times \left(e^{y\sqrt{-1}} + e^{-y\sqrt{-1}}\right) - \tfrac{1}{2} \times \frac{e^{-x\sqrt{-1}}}{2\sqrt{-1}} \times \left(e^{y\sqrt{-1}} + e^{-y\sqrt{-1}}\right)$$

$$= \tfrac{1}{2} \times \frac{e^{(x+y)\sqrt{-1}} + e^{(x-y)\sqrt{-1}}}{2\sqrt{-1}} - \tfrac{1}{2} \times \frac{e^{-(x-y)\sqrt{-1}} + e^{-(x+y)\sqrt{-1}}}{2\sqrt{-1}}$$

$$= \tfrac{1}{2} \times \frac{e^{(x+y)\sqrt{-1}} - e^{-(x+y)\sqrt{-1}}}{2\sqrt{-1}} - \tfrac{1}{2} \times \frac{e^{(x-y)\sqrt{-1}} - e^{-(x+y)\sqrt{-1}}}{2\sqrt{-1}}$$

$= \frac{1}{2} \times$ sin. $(x + y) + \frac{1}{2}$ sin. $(x - y)$.

2. $\overline{\text{cos.}\, x}^{\,n} = \frac{1}{2^{n-1}}\left(\text{cos.}\, nx + n \,.\, \text{cos.}\, (n-2)\,x + \frac{n\,.\,(n-1)}{2}\,\text{cos.}\,(n-4)\,x + \&c.\right)$

$+ \frac{1}{2^{n-1}} \times \frac{(n+3)\,(n+5)\,(n+7)\ldots\ldots 2n}{(n-1)\,(n-3)\,(n-5)\ldots\ldots 5\cdot 3}$ cos. (x) (n an odd number) or

$= \frac{1}{2^{n-1}}\left(\text{cos.}\, nx + n \,.\, \text{cos.}\, (n-2)\,x + \frac{n\,.\,(n-1)}{2}\,\text{cos.}\,(n-4)\,x + \&c.\right)$

$+ \frac{(n+2)\,(n+4)\ldots 2n}{n\,.\,(n-2)\ldots 4\ \ 2} \times \frac{1}{2}$, n being an even number:

for cos. $x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}$ $\therefore \overline{\text{cos.}\, x}^{\,n} = \left(\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}\right)^n$

$= \left(\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}\right)^{n-1} \times \left(\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}\right)$; now,

if $\left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right)^{n-1}$ were $= e^{(n-1)x\sqrt{-1}} + (n-1)$ $e^{(n-3)x\sqrt{-1}} + \&c.$ $\left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right)^n$ would $= e^{nx\sqrt{-1}}$ $+ n\, e^{(n-2)x\sqrt{-1}} + \&c.$ or, if the developement of $\left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right)^{n-1}$ were according to the law of the binomial theorem, the developement of $\left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right)^n$ would be according to the same law; but the developement of

$\left(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}\right)^2$ is according to that law $\therefore$ of $\left(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}\right)^3$ $\therefore$ &c. $\therefore$ $\left(\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}\right)^n$

$$= \frac{1}{2^n}\left(e^{nx\sqrt{-1}} + n e^{(n-2)x\sqrt{-1}} + \frac{n(n-1)}{2} e^{(n-4)x\sqrt{-1}} + \text{\&c.}\right.$$

$$+ \frac{(n+3)(n+5)\ldots\ldots 2n}{(n-1)(n-3)\,n-5\cdots 3} \times \left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right) + \text{\&c.}$$

$$\left. + n e^{-(n-2)x\sqrt{-1}} + e^{-nx\sqrt{-1}}\right) \;(n \text{ odd}), \text{ or}$$

$$= \frac{1}{2^{n-1}}\left(\frac{e^{nx\sqrt{-1}}+e^{-nx\sqrt{-1}}}{2}\right) + n \times \left(\frac{e^{(n-2)x\sqrt{-1}}+e^{-(n-2)x\sqrt{-1}}}{2}\right) + \text{\&c.}$$

$$+ \frac{(n+3)(n+5)\ldots\ldots 2n}{(n-1)(n-3)\,n-5\cdots 3} \times \left(\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}\right)$$

$$= \frac{1}{2^{n-1}}\left(\cos.\, nx + n \,.\, \cos.\,(n-2)\,x + \text{\&c.} + \frac{(n+3)(n+5)\ldots\ldots 2n}{(n-1)(n-3)\ldots\ldots 3}\cos.\, x\right)$$

when n is even

$$\left(\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}\right)^n = \frac{1}{2^n}\times\left(e^{nx\sqrt{-1}} + n e^{(n-2)x\sqrt{-1}}\right.$$

$$\left. + \frac{n\,.\,(n-1)}{2} e^{(n-4)x\sqrt{-1}} + \text{\&c.} + \frac{(n+2)(n+4)\ldots\ldots 2n}{n\,.\,(n-2)(n-4)\ldots 4\,.\,2}\right)$$

$$= \frac{1}{2^{n-1}}\times\left(\frac{e^{nx\sqrt{-1}}+e^{-nx\sqrt{-1}}}{2}\right) + n\times\left(\frac{e^{(n-2)x\sqrt{-1}}+e^{-(n-2)x\sqrt{-1}}}{2}\right) + \text{\&c.}$$

$$+ \frac{(n+4)(n+6)\ldots 2n}{(n-2)(n-4)\ldots\ldots 2} \times \left(\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}\right) + \frac{(n+2)(n+4)\ldots\ldots 2n}{n\,.\,(n-2)-1\ldots 2}\times\frac{1}{2}$$

$$= \frac{1}{2^{n-1}}\times\left(\cos.\, nx + n\,.\,\cos.\,(n-2)\,x + \text{\&c.}\,\frac{(n+4)(n+6)\ldots\ldots 2n}{(n-2)(n-4)\ldots\ldots 2}\cos.\, 2x\right.$$

$+ {}^{*}\,\frac{(n+2)(n+4)\ldots\ldots 2n}{n\,.\,(n-2)\ldots\ldots 2}\times\frac{1}{2}$. By a similar investigation, $\overline{\text{sine}\, x}^{\,n}$ may be determined.

* This is the greatest coefficient in the binomial, and belongs to the term in which the index of e is o; for the coefficient of the mth term is $\frac{n(n-1)(n-2)\ldots(n-m+2)}{1\,.\,2\,.\,3\quad\ldots m-1}$.

3. The sum of $\cos. x + \cos. 2x + \cos. 3x \ldots + \cos. nx$ $= \frac{\cos. nx - \cos. (n+1)x + \cos. x - 1}{2(1-\cos. x)}$, for this sum is

$$\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} + \frac{e^{2x\sqrt{-1}} + e^{-2x\sqrt{-1}}}{2} + \&c. \ldots \frac{e^{nx\sqrt{-1}} + e^{-nx\sqrt{-1}}}{2},$$

or $\frac{1}{2} \times \left(e^{x\sqrt{-1}} + e^{2x\sqrt{-1}} + \ldots\ldots\ldots + e^{nx\sqrt{-1}}\right)$
$+ \frac{1}{2} \times \left(e^{-x\sqrt{-1}} + e^{-2x\sqrt{-1}} + \&c. \ldots\ldots\ldots e^{-nx\sqrt{-1}}\right).$

Now, according to the explanation that has been given of equality,

$$e^{x\sqrt{-1}} + e^{2x\sqrt{-1}} \ldots e^{nx\sqrt{-1}} = e^{x\sqrt{-1}} \times \left(1 + e^{x\sqrt{-1}} + e^{2x\sqrt{-1}} \ldots e^{\overline{n-1}\,x\sqrt{-1}}\right);$$

if S were an abridged symbol, which, developed according to a certain form, became the series $e^{x\sqrt{-1}} + e^{2x\sqrt{-1}} \ldots e^{nx\sqrt{-1}}$, then $1 + S - e^{nx\sqrt{-1}}$ would truly represent $1 + e^{x\sqrt{-1}} + e^{2x\sqrt{-1}} \ldots e^{(n-1)x\sqrt{-1}}$, and therefore S and $e^{x\sqrt{-1}} \times \left(1 + S - e^{nx\sqrt{-1}}\right)$ would be expressions equally significant, or S would $= e^{x\sqrt{-1}} + Se^{x\sqrt{-1}} - e^{(n+1)x\sqrt{-1}}$, or $S = \frac{e^{(n+1)x\sqrt{-1}} - e^{x\sqrt{-1}}}{e^{x\sqrt{-1}} - 1}$, the form that must be given to S; so that, when expanded after a known form, it becomes $e^{x\sqrt{-1}} + e^{2x\sqrt{-1}} \ldots\ldots e^{nx\sqrt{-1}}$.

In like manner, the abridged symbol for $e^{-x\sqrt{-1}} + e^{-2x\sqrt{-1}} \ldots e^{-nx\sqrt{-1}}$

but index $n - 2m + 2 = 0 \therefore 2m = n + 2 \therefore$ the coefficient is $\frac{n.(n-1)\ldots\left(\frac{n+4}{2}\right)\frac{(n+2)}{2}}{1.2.3\ldots\frac{(n-2)}{2}\frac{n}{2}}$ $= \frac{(n+2)(n+4)\ldots\ldots(2n-2)2n}{n.(n-2)(n-4)\ldots\ldots 4.2}$; and it is the greatest coefficient, since the coefficients of the adjacent terms are determined by multiplying it by $\frac{m-1}{n-m+2}$ and $\frac{n-m+1}{m}$ respectively; or, since $m = \frac{n+2}{2}$, by $\frac{n}{n+2}$ and $\frac{n}{n+2}$.

is $\frac{e^{-(n+1)x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{-x\sqrt{-1}} - 1}$. Hence the series is represented

by $\frac{1}{2}\left(\frac{e^{(n+1)x\sqrt{-1}} - e^{x\sqrt{-1}}}{e^{x\sqrt{-1}} - 1}\right) + \frac{1}{2}\frac{e^{-(n+1)x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{-x\sqrt{-1}} - 1}$;

which is the same as

$$\frac{1}{2} \times \frac{e^{(n+1)x\sqrt{-1}} - e^{x\sqrt{-1}}}{e^{x\sqrt{-1}} - 1} \times \frac{e^{-x\sqrt{-1}} - 1}{e^{-x\sqrt{-1}} - 1} + \frac{1}{2}\frac{e^{-(n+1)x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{-x\sqrt{-1}} - 1}$$

$$\times \frac{e^{x\sqrt{-1}} - 1}{e^{x\sqrt{-1}} - 1}, \text{ or is}$$

$$= \frac{1}{2} \times \frac{e^{nx\sqrt{-1}} - 1 - e^{(n+1)x\sqrt{-1}} + e^{x\sqrt{-1}}}{2 - e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} \cdot \cdot \cdot \cdot \cdot \cdot$$

$$+ \frac{1}{2} \times \frac{e^{-nx\sqrt{-1}} - 1 - e^{-(n+1)x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2 - e^{x\sqrt{-1}} + e^{x\sqrt{-1}}}$$

$$= \frac{1}{2}\frac{\left(\frac{e^{nx\sqrt{-1}} + e^{-nx\sqrt{-1}}}{2}\right) - \left(\frac{e^{x(n+1)\sqrt{-1}} + e^{-(n+1)x\sqrt{-1}}}{2}\right) + \left(\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}\right) - 1}{2 \times \left(1 - \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}\right)}$$

$$= \frac{\cos. nx - \cos. (n+1)x + \cos. x - 1}{2 \times (1 - \cos. x)}.$$

In like manner, the series $\sin. x + \sin. 2x + \ldots \sin. mx$ may be shewn $= \frac{\sin. x - \sin. (m+1)x + \sin. mx}{2(1 - \cos. x)}$. The sum of $\overline{\cos. x}^2$ $+ \overline{\cos. 2x}^2 \ldots \overline{\cos. nx}^2$ may easily be found, by expressing it under the form $\left(\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}\right)^2 + \left(\frac{e^{2x\sqrt{-1}} + e^{-2x\sqrt{-1}}}{2}\right)^2$ $+ \ldots \left(\frac{e^{nx\sqrt{-1}} + e^{-nx\sqrt{-1}}}{2}\right)^2$; which, by expanding the terms,

becomes $\left(\frac{e^{2x\sqrt{-1}}+2+e^{-2x\sqrt{-1}}}{4}\right)+\left(\frac{e^{4x\sqrt{-1}}+2+e^{-4x\sqrt{-1}}}{4}\right)+$ &c.

and, consequently, equals $\frac{n}{2}+\frac{1}{2}\cos.(2x)+\cos.(4x)+\cos.(6x)\dots\cos.(2nx)$, or $\frac{n}{2}+\frac{1}{2}\,\frac{\cos.2nx-\cos.(2n+2)x+\cos.2x-1}{2(1-\cos.2x)}$.

Similarly may be determined the sums of $\overline{\sin.x}^2+\overline{\sin.2x}^2+\dots\overline{\sin.nx}^2$, of $\overline{\cos.x}^3+\overline{\cos.2x}^3+\dots\overline{\cos.nx}^3$, and generally of $\overline{\cos.x}^n+\overline{\cos.2x}^n+\dots\overline{\cos.mx}^n$; for, ($n$ being even,* $A=\frac{(n+2)(n+4)(n+6)\dots 2n}{n\,(n-2)(n-4)\dots 4.2}$, and $B=\frac{(n+4)(n+6)\dots 2n}{(n-2)(n-4)\dots 4.2}$)

$$\overline{\cos.x}^n=\frac{1}{2^{n-1}}\left(\cos.nx+n.\cos.(n-2)x+\text{\&c.}\dots B\cos.2x+\frac{A}{2}\right)$$

$$\overline{\cos.2x}^n=\frac{1}{2^{n-1}}\left(\cos.2nx+n.\cos.2(n-2)x+\text{\&c.}\dots B\cos.4x+\frac{A}{2}\right)\ \text{\&c.}$$

$$\overline{\cos.mx}^n=\frac{1}{2^{n-1}}\left(\cos.mnx+n.\cos.m(n-2)x+\text{\&c.}\dots B.\cos.2mx.+\frac{A}{2}\right)$$

* The coefficient $A=\frac{(n+2)(n+4)(n+6)\dots 2n}{n.(n-2)(n-4)\dots 4.2}$ may be differently expressed, thus,

$$A=\frac{n.(n-1)(n-2)\dots\dots n-m+2}{(m-1)(m-2)(m-3)\dots 4.3.2.1}\quad(n-2m+2=0)$$

$$=\frac{\left(\frac{n}{2}+\frac{n}{2}\right).(n-1).\left(\frac{n}{2}-1+\frac{n}{2}-1\right).(n-3).\left(\frac{n}{2}-2+\frac{n}{2}-2\right)\dots\left(\frac{n}{2}+1\right)}{\frac{n}{2}.\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\dots\dots\frac{n}{4}\left(\frac{n}{1}-1\right)\left(\frac{n}{4}-2\right)\dots 4.3.2.1}$$

$$=\frac{2(n-1)\,2(n-3)\,2(n-5)\dots\dots 2\left(\frac{n}{2}+1\right)}{\frac{n}{4}.\left(\frac{n}{4}-1\right)\left(\frac{n}{4}-2\right)\dots\dots 4.3.2.1}$$

$$=\frac{2^{\frac{n}{4}}(n-1)(n-3)(n-5)\dots\left(\frac{n}{2}+1\right)}{\frac{n}{4}\left(\frac{n}{4}-1\right)\left(\frac{n}{4}-2\right)\dots 4.3.2.1}=\frac{2^{\frac{n}{2}}(n-1)(n-3)\dots\left(\frac{n}{2}+1\right)}{\frac{n}{2}\left(\frac{n}{2}-2\right)\left(\frac{n}{2}-4\right)\dots 8.6.4.2}$$

$$=\frac{2^{\frac{n}{2}}(n-1)(n-3)(n-5)\dots\frac{n}{2}+1}{\frac{n}{2}\left(\frac{n}{2}-2\right)\left(\frac{n}{2}-4\right)\dots 6.4.2}\times\frac{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-3\right)\left(\frac{n}{2}-5\right)\dots 5.3.1}{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-3\right)\left(\frac{n}{2}-5\right)\dots 5.3.1}$$

$$=\frac{2^{\frac{n}{2}}(n-1)(n-3)\dots\dots 5.3.1}{\frac{n}{2}\left(\frac{n}{2}-1\right)\dots\dots 6.5.4.3.2.1},\ \text{or}\ \frac{1.3.5\dots(n-1)}{1.2.3\dots\frac{n}{2}}2^{\frac{n}{2}}.$$

$\therefore$ adding the quantities that are vertical to each other, the series

$$= \frac{1}{2^{n-1}} \left(\cos. n x \quad + \cos. 2 n x \quad + \&c. \ldots \cos. m n x \right)$$
$$+ \frac{n}{2^{n-1}} \left(\cos. (n-2) x + \cos. 2 (n-2) x + \&c. \ldots \cos. m (n-2) x \right)$$
$$+ \&c.$$
$$+ \frac{B}{2^{n-1}} \left(\cos. 2 x \quad + \cos. 4 x \quad + \&c. \ldots \cos. 2 m x \right)$$
$$+ \frac{m A}{2^{n-1}} \left(\frac{1}{2} \right).$$

Now, each horizontal row consists of a series of cosines of arcs in arithmetical progression; and the sum of each series may immediately be obtained from the expression deduced in proposition 3d.

I think it superfluous to give more examples, since the object of this memoir is rather to shew the logical justness of a method, than its commodiousness or extent: all other propositions relative to lines drawn in a circle, when expressed by aid of the symbol $\sqrt{-1}$, the same principle of explanation regulates; the principle once understood, the operations become mechanical, require attention, but are attended with no real mental difficulty.

It is inaccurate to call $\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}$ an imaginary value of the cosine of an arc: the expression expanded is a real one. By use of the symbol $\sqrt{-1}$, and of the forms proved to obtain in the combination of real quantities, a mode of notation is obtained, by which we may express sines and cosines, &c. relatively to their arc.

If the process by which the foregoing propositions have been established require illustration, I would ask what demonstration is, when the characters employed are signs of ideas, or repre-

sentatives of real things; and demonstration would be defined to be, a method of shewing the agreement of remote ideas by a train of intermediate ideas, each agreeing with that next it; or, in other words, a method of tracing the connection between certain principles and a conclusion, by a series of intermediate and identical propositions, each proposition being converted into its next, by changing the combination of signs that represent it, into another shewn to be equivalent to it.

Exactly according to this plan have the foregoing propositions been demonstrated: the symbol for the sine of x is $\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$, for the cosine of y is $\frac{e^{y\sqrt{-1}} + e^{-y\sqrt{-1}}}{2}$, and the connection was traced between $\left(\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}\right)$ $\times \left(\frac{e^{y\sqrt{-1}} + e^{-y\sqrt{-1}}}{2}\right)$ and $\frac{1}{2} \times \frac{e^{(x+y)\sqrt{-1}} - e^{-(x+y)\sqrt{-1}}}{2\sqrt{-1}}\Big) + \frac{1}{2}$ $\times \frac{e^{(x-y)\sqrt{-1}} - e^{-(x-y)\sqrt{-1}}}{2\sqrt{-1}}$, by a series of transformations, each of which was shewn to be lawful, by referring to what $e^{x\sqrt{-1}}$ &c. was made to represent, and to the nature of the operations directed to be performed by the signs $\times$, $+$, &c. thus, the transformation of $e^{x\sqrt{-1}} \times \left(e^{y\sqrt{-1}} + e^{-y\sqrt{-1}}\right)$ into $e^{x\sqrt{-1}} \times e^{y\sqrt{-1}}$ $+ e^{x\sqrt{-1}} \times e^{-y\sqrt{-1}}$ is lawful, because the same series results, whether $e^{y\sqrt{-1}}$ and $e^{-y\sqrt{-1}}$ be first expanded, and then each term of their sum be combined with $e^{x\sqrt{-1}}$, or whether $e^{x\sqrt{-1}}$ be separately combined with each term of the developements for $e^{y\sqrt{-1}}$ and $e^{-y\sqrt{-1}}$, and then the resulting terms added together: again, the transformation of $e^{x\sqrt{-1}} \times e^{y\sqrt{-1}}$ into $e^{(x+y)\sqrt{-1}}$ is lawful, because the same series results, whether

$e^{x\sqrt{-1}}$ and $e^{y\sqrt{-1}}$ be expanded, and then their terms combined according to the rules for the multiplication of quantities, or whether $e^{(x+y)\sqrt{-1}}$ be immediately expanded, by writing $(x+y)\sqrt{-1}$ for x, in the series for e^x.

The other demonstrations examined will appear conducted on the same principle, which is simple, and of easy and immediate application: hence, although the symbol $\sqrt{-1}$ be beyond the power of arithmetical computation, the operations in which it is introduced are intelligible, and deserve, if any operations do, the name of reasoning.

It is almost superfluous to observe, that if the operations by means of imaginary symbols have appeared to be necessarily true, the arguments founded on the analogy subsisting between the circle and hyperbola must be abandoned, as unsatisfactory. What has been proved concerning the properties of lines appertaining to a circle, has been so without any mention of the hyperbola; and I may say, without danger of refutation, that the demonstrations would be strictly true, if such a curve as the hyperbola had never been invented. Add to this, that imaginary expressions are useful in leading to just conclusions, in investigations purely algebraical.

The chief purpose of this Paper is fulfilled, if it has appeared that the operations with imaginary symbols possess the evidence and rigour of mathematical demonstration: whether it is convenient to use imaginary quantities in analytical investigation, must be determined on the grounds of abridgment and extensive application. In the cases that I have considered, imaginary expressions are not, I know, indispensably necessary: they are excluded from each of three different methods for the solution of propo-

sitions relative to lines belonging to a circle, given by M. LAGRAVE, by EULER, (Introductio in Analysin Infinitorum, p. 198.) and by BOSSUT. (Mem. de l'Acad. 1769, p. 453.) I am, however, of opinion, that the method of representing sines, cosines, &c. by their abridged algebraical symbols, (such as is given in this Paper,) is the most easy and extensive in its application.*

It will be consistent with the purpose of the present memoir, to consider some of the expressions which I imagine are alluded to, by those who complain of the abuses, paradoxes, &c. introduced by negative and impossible quantities.

The quantity $\frac{4 \log. \sqrt{-1}}{\sqrt{-1}}$, which JOHN BERNOUILLI proved to be the circumference of a circle, is merely an abridged symbol, founded on a form proved for real quantities: the sense in which it is to be understood is this, that if in the series for log. x, viz. $(x - x^{-1}) - \frac{1}{2}(x^2 - x^{-2}) + \frac{1}{3}(x^3 - x^{-3}) - \&c.$ $\sqrt{-1}$ is substituted for x, and the terms multiplied by 4 and divided by $\sqrt{-1}$, the resulting series expresses the circumference of a circle.

The expressions

$$(1)\ \sin.(a + b\sqrt{-1}) = \tfrac{1}{2}(e^b + e^{-b})\sin. a + \tfrac{\sqrt{-1}}{2}(e^b - e^{-b})\cos. a,$$

$$(2)\ \cos.(a + b\sqrt{-1}) + \cos.(a - b\sqrt{-1}) = (e^b + e^{-b})\cos. a,$$

are due to EULER: the sense in which alone they are to be understood is this, that the series which results from substituting

* M. BOSSUT does not sum any series beyond that of the fourth power of sines and cosines of arcs in arithmetical progression: he contents himself with saying, that the general law for $\overline{\cos. q}^n + \overline{\cos. 2q}^n$ &c. may easily be discovered.

$a + b\sqrt{-1}$ for x, in the series $x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5}$ &c. proved for the sine of an arc x, is the same as what results from expanding e^b, e^{-b}, cos. a, sin. a, &c. and combining the terms after the manner directed by the signs $+$, $\times$, &c. A like explanation is to be given of the second expression.

If m be an integer, c the semi-circumference, and $a = \left(\frac{4m \pm 1}{2}\right)c$, then cos. $a = 0$, and the first expression becomes sin. $(a + b\sqrt{-1}) = \frac{1}{2}(e^b + e^{-b})$ sin. a. According to the explanation I have given, this expression is very perspicuous and intelligible; but Euler, inattentive to its true meaning, gives it an air of mystery and paradox, when he says that an impossible arc may have a real sine.

The symbol $(\sqrt{-1})^{\sqrt{-1}}$, Euler proved equal to 0.20787957, &c. To understand its meaning, we must recur to the form from which it was derived: now, according to the definition that has been given of equality between imaginary expressions, it may be shewn that

$$(a+b\sqrt{-1})^{m+n\sqrt{-1}} = r^m e^{-nx}\left(\cos.(mx+n \times l.r) + \sqrt{-1}\sin.(mx+n \times l.r)\right)$$

r being $= \sqrt{a^2 + b^2}$, sin. $x = \frac{b}{r}$, cos. $x = \frac{a}{r}$.

Now, if a be put $= 0$, $m = 0$, $b = 1$, $n = 1$, the expression $(a + b\sqrt{-1})^{m + n\sqrt{-1}}$ becomes $(\sqrt{-1})^{\sqrt{-1}}$, and the expression to which it is equal becomes $e^{-\frac{c}{4}}$ (c circumference). Or the meaning of the symbol may be thus explained, x^x is the same as $e^{x \log. x}$ $\therefore$ if $\sqrt{-1}$ be put for x, $(\sqrt{-1})^{\sqrt{-1}} = e^{\sqrt{-1}\log.\sqrt{-1}}$; but it has appeared that log. $\sqrt{-1}$ is an abridged symbol for

$\frac{c}{4\sqrt{-1}}$; hence $(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{c}{4}}$, or $(\sqrt{-1})^{\sqrt{-1}}$, is an abridged symbol for the series $1 - \frac{c}{4} + \frac{c^2}{1 \cdot 2 \cdot 4^2} - \frac{c^3}{1 \cdot 2 \cdot 3 \cdot 4^3} + \&c.$

I do not pretend to say, that such expressions as the above are likely to occur in investigation, and to be practically useful; my sole concern is to shew, that they are perfectly intelligible, and the necessary consequences of certain assumptions.

The paradoxes and contradictions mutually alleged against each other, by mathematicians engaged in the controversy* concerning the application of logarithms to negative and impossible quantities, may be employed as arguments against the use of those quantities in investigation. The paradoxes and contradictions will quickly disappear, by adopting the same mode of explanation that has been already employed in this paper. The memoir of EULER is in some parts erroneous, and frequently unsatisfactory.

The use of a mathematical definition is, to deduce from it the properties of the thing defined; and, whatever definition of logarithms be taken, we either have immediately, or may deduce for the purpose of computation, an expression such as $y = e^x$, in which x is the logarithm of e^x to the base e; the developement of e^x has been proved to be $1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \frac{A^3 x^3}{1 \cdot 2 \cdot 3} + \&c.$ A being $= (e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 - \&c.$

* This controversy exercised a long time the abilities of LEIBNITZ, BERNOUILLI, EULER, D'ALEMBERT, and FONCENEX. The Commercium Philosophicum et Mathematicum, published at Lausanne, in 1745, and containing the letters of the two first controvertists, I have never seen; but I presume, that all the essential arguments of the controversy are to be found in FONCENEX's Memoir, (Vol. I. Mem. de Turin,) in EULER's, (Mem. de Berlin, 1749.) and in D'ALEMBERT's Opuscules, Vol. I.

Now the question concerning the logarithms of negative quantities, in a precise form, and freed from its verbal ambiguities, is this; is the symbol which, substituted for x in the developement of e^x, makes y or $e^x = -1$, the sign of a real quantity or not?

In the expression e^x, x is the logarithm of e^x, and, by extension, $x\sqrt{-1}$ is to be called the logarithm of $e^{x\sqrt{-1}}$. Now $e^{x\sqrt{-1}}$ is the symbol for $1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3}$ &c. or $\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \text{\&c.}\right) + \sqrt{-1}\left(x - \frac{x^3}{1.2.3} + \text{\&c.}\right)$ or $e^{x\sqrt{-1}}$ may be said to be $= \cos. x + \sqrt{-1}\sin. x$. Hence, $x\sqrt{-1}$ is the logarithm of $\cos. x + \sqrt{-1}\sin. x$; when the arc x is equal 0, or 2π, or 4π, or 6π, or generally $2m\pi$, its $\cos. x = 1$.

Hence 0 is log. 1,
or $2\pi\sqrt{-1}$ is log. 1,
or $4\pi\sqrt{-1}$, or generally $2m\pi\sqrt{-1}$, is log. 1.

Hence, if $y = 1$, the equation $y = e^x$ becomes $1 = e^{2m\pi\sqrt{-1}}$, (m being any number of the progression $0, 1, 2, 3, 4$, &c.)

Again, if the arc x is equal π, or 3π, or 5π, or generally $(2m + 1)\pi$, its $\cos. x = -1$.

$\therefore$ of -1, either $\pi\sqrt{-1}$, or $3\pi\sqrt{-1}$, or $5\pi\sqrt{-1}$, or $(2m+1)\pi\sqrt{-1}$, is to be called the logarithm; hence, if $y = -1$, the equation $y = e^x$ becomes $-1 = e^{(2m+1)\pi\sqrt{-1}}$.

The meaning of the logarithms of 1 and -1 are then thus to be understood. If in the series

$1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} +$ &c. for x be substituted

either $0, 2\pi\sqrt{-1}, 4\pi\sqrt{-1}, 6\pi\sqrt{-1} \ldots$ or $(2m\pi\sqrt{-1})$, the equation $1 = 1 + x + \frac{x^2}{1.2}$ &c. becomes identical; and if, in the same equation, for x be substituted either $\pi\sqrt{-1}$, $3\pi\sqrt{-1}$, $5\pi\sqrt{-1}$, ... or $(2m+1)\pi\sqrt{-1}$, the equation $-1 = 1 + x + \frac{x^2}{1.2} +$ &c. becomes identical; for, substitute $2m\pi\sqrt{-1}$ for x in series for e^x, then it becomes

$$1 + (2m\pi\sqrt{-1}) - \frac{(2m\pi)^2}{1.2} - \frac{(2m\pi)^3\sqrt{-1}}{1.2.3} + \frac{(2m\pi)^4}{1.2.3.4},$$

or $1 - \frac{(2m\pi)^2}{1.2} + \frac{(2m\pi)^4}{1.2.3.4} -$ &c. $+ \sqrt{-1}\left(2m\pi - \frac{(2m\pi)^3}{1.2.3} + \text{\&c.}\right)$ or $\cos. 2m\pi + \sqrt{-1}\sin. 2m\pi$; but $\sin. 2m\pi = 0$ $\cos. 2m\pi = 1 \therefore 1 = 1$.

In like manner it will appear, that $-1 = -1$, if $(2m+1)\pi\sqrt{-1}$ be substituted for x.

Since $1 = e^{2m\pi\sqrt{-1}}$ $\quad (1)^2 = e^{4m\pi\sqrt{-1}}$, and since $-1 = e^{(2m+1)\pi\sqrt{-1}}$ $\quad (-1)^2 = e^{(2m+1)2\pi\sqrt{-1}}$; but, in the developement of $e^{(2m+1)2\pi\sqrt{-1}}$, the rule of $- \times - = +$ is observed $\therefore$ it must be observed on the other side of the equation; hence, if $(1)^2 = (-1)^2$, the only strict conclusion that can be drawn is this, that $e^{4m\pi\sqrt{-1}}$ and $e^{(2m+1)2\pi\sqrt{-1}}$* developed, produce the same series. It is a false consequence that, since $(1)^2 = (-1)^2$ the logarithm of $(1)^2 =$ logarithm $(-1)^2$, the logarithms are the indices $4m\pi\sqrt{-1}$ and $(2m+1)2\pi\sqrt{-1}$.

* 1 is equal $e^{2\pi\sqrt{-1}}$ or $e^{4\pi\sqrt{-1}}$ &c. or $e^{2\pi\sqrt{-1}}$, and $e^{4\pi\sqrt{-1}}$ expanded are each $= 1$; no consequence can be drawn from what is true for quantities elevated to real powers, since $e^{x\sqrt{-1}}$ can only have the meaning assigned it, that of being an abridged symbol.

EULER, confounding the common meaning of logarithms with their scientific definition, granted that the log. $(1)^2$ was equal log. $(-1)^2$, and endeavoured to reconcile the contradictions that immediately followed from such a concession.

The arguments intended to prove that the logarithms of negative quantities were real, may easily be shewn to be nugatory. EULER, certainly too much attached to mere calculation, instead of directly opposing them, sought to divert their force. D'ALEMBERT asserted, that the two progressions 1, 2, 3, &c. $-1, -2, -3$, &c. might have the same series of logarithms, o, p, q, r, &c. This is true, if -2 means $2 \times (-1)$, -3, $3 \times (-1)$, &c. or the progression $-1, -2, -3$, &c. is the same as $1 \times (-1)$, $2(-1)$, $3(-1)$, &c. wherein (-1) is considered as an unit, or as (x) a sign of a real quantity. But the question is thus evaded; since $-1, -2, -3$, $-$, &c. is brought precisely under the same predicament as 1, 2, 3, 4, &c. The only real point of inquiry could be, whether, consistently with the system of logarithms established for positive quantities, the logarithms of negative quantities were real.

A second argument brought by BERNOUILLI and D'ALEMBERT was, that since $a : -a :: -a : a \therefore (a)^2 = (-a)^2 \therefore$ log. $(a)^2$ = log. $(-a)^2 \therefore 2$ log. $(+a) = 2$ log. $(-a) \therefore$ log. (a) = log (a). This proposition, affirmed by D'ALEMBERT to be strictly true, viz. $(a : -a :: -a : a)$, was granted to be so by EULER, although it ought to have been denied; since, thus abstractedly proposed, it is absurd and unintelligible, and impossible to be proved. If, however,* $(-a)^2$ be assumed $= (a)^2$, $(-1)^2 =$

* I have explained, in a former note, for what reasons, and in what circumstances, $-a \times -a$ is the same as $a \times a$.

$(1)^4$, then the equality log. $(-a)^2 = \log. (a)^2$ becomes intelligible; since it means that the measure of the ratio between $(-a)^2$ and $(-1)^2$ is equal the measure of the ratio between $(a)^2$ and $(-1)^2$; but then this argument becomes the same as the former, and is equally illusory; for $-a$ and -1 are in fact made a and 1. If logarithms be defined, the measures of ratios existing between real quantities, then it is absurd to attempt deducing the logarithms of negative quantities from any reasoning on the relation that 1 has to -1; since there is no necessary connection between 1 and -1; and, independently of certain assumptions, the ratio of $1 : -1$ is perfectly unintelligible. Indeed the question admits no other meaning than that I originally assigned it: if a form demonstrated for positive quantities be extended, then certain symbols may be exhibited, which, agreeably to such extension, are called the logarithms of negative quantities.

Other arguments than those I have mentioned, were drawn from the theories of curves and fluxions, not only foreign to the question, which was purely algebraical, but of small weight; had they been of greater, the inquiry would necessarily have been diverted on the nature of the connection existing between these theories and algebra.

In this controversy, the predominancy of the "Esprit Geometrique" is remarkable; if, in an inquiry purely mathematical, any ambiguity or paradox presents itself, the most simple and natural method is, to recur to the original notions on which calculation has been founded. Instead of pursuing this method, the controvertists sought to derive illustration from obscure doctrines, or to discover the latent truth amidst the complex forms and involutions of analysis.

My inquiry concerning impossible quantities, has been confined to their use in representing lines belonging to the circle, and to the necessary truth of the conclusions obtained by their means; led by the connection of the subjects, I have made a small deviation, to examine the true meaning of certain symbols, and the contradictions said to embarrass the doctrines of logarithms when applied to negative quantities. The use, however, of impossible quantities has been extended to all parts of analysis. By their aid are determined, the values of formulas that occur in the science of the motion of fluids, the numerators of partial fractions as $\left(\frac{Ax + B}{x^2 + 2\alpha x + \alpha^2 + \beta^2}\right)$, the developement of forms as $(r^2 - 2rr' \cos. z + r'^2)^{-m}$, and the integration of many differential equations.* If, in these cases, the operation

* By means of impossible quantities, CARDAN's rule, in the irreducible case, when the three roots are real and incommensurable, may be applied. In the equation $x^3 - px + r$, when $\frac{r^2}{4}$ is $< \frac{p^3}{27}$, the root appears under the form $\sqrt[3]{a + b\sqrt{-1}} + \sqrt[3]{a \mp b\sqrt{-1}}$; and, by putting $a + b\sqrt{-1} = r(\cos. x^1 + \sin. x^1 . \sqrt{-1})$, the three roots may easily be shewn to be $2\sqrt{\frac{p}{3}} \cos. \frac{x^1}{3}$, $2\sqrt{\frac{p}{3}} \cos. \left(\frac{x^1 + 2c}{3}\right)$, $2\sqrt{\frac{p}{3}} \cos. \overline{\left(\frac{4c + x^1}{3}\right)}$.

This method, indeed, only exhibits the linear value of the root; the algebraic value cannot generally be exhibited. In some particular cases, the algebraic value may be obtained, when the series that results from adding the terms of the developements of $\sqrt[3]{a + b\sqrt{-1}}$ and $\sqrt[3]{a - b\sqrt{-1}}$ can be summed; as M. NICOLE (Mem. de l'Acad. 1738, pages 97, 244,) has shewn, who first proved the expression $\sqrt[3]{a + b\sqrt{-1}} + \sqrt[3]{a - b\sqrt{-1}}$, when expanded, to be real.

I am of opinion, that CARDAN's solution, in the irreducible case, cannot be extended so as to obtain the general linear value, or in particular cases the algebraic value, except by operations with impossible quantities; and that when, by aid of impossible quantities, the general linear value or particular algebraic values are exhibited, such

with imaginary symbols are intelligible and just, the only argument for their exclusion must be founded on the existence of methods more general and expeditious.

The application of imaginary quantities to the theory of equations, has perhaps been made more extensively than to any other part of analysis. To consider the propriety of this application on the grounds of perspicuity and conciseness, a long discussion would be necessary. I may, however, be here permitted merely to state my opinion, that impossible quantities must be employed in the theory of equations, in order to obtain general rules and compendious methods. The demonstration of the principal proposition, that every root of an equation is comprised under the form $M + N\sqrt{-1}$, and that consequently every equation of $2n$ dimensions, is always divisible into n quadratic factors, appears to me, I confess, deficient in evidence and mathematical rigour. To establish this proposition, and to prove likewise, that every imaginary expression derived from transcendental operations is always comprised under the form $M + N\sqrt{-1}$, is the object of two Memoirs by D'ALEMBERT and EULER. (Mem. de Berlin, 1746, 1749.)

M. FONCENEX, (Mem. de Turin.) LAGRANGE, (Mem. de Berlin, 1771, 1772, 1773,) LAPLACE, WARING, and other mathematicians, have directed their inquiries towards the same subject.*

values are necessarily and by strict consequence true; and not true because they may be verified by a distinct or more rigorous investigation, nor because the operations have a tacit and implied reference to other more legitimate operations.

* None of the demonstrations go farther than to shew the *possibility* of resolving an equation of $2n$ dimensions into n quadratic factors. The actual resolution of equations that pass the fourth degree, has not hitherto been executed. Of the labours of such learned men as those I have mentioned, I speak with the greatest diffidence; the mere knowledge, however, of the possibility of the resolution of equations, appears to

The nature of the subject has obliged me to give this paper, in several of its parts, somewhat of a controversial cast: for having used the freedom of discussion in matters of pure science, an apology is unnecessary; the memoir of the ingenious person whose opinion I have formally controverted, I can most sincerely commend for every thing, except the justness of the principle of explanation.

To excuse the prolixity that may appear in the explanation of the operations, and in the proofs of their justness, I wish it to be considered, that it was necessary to examine the notions on which calculation ultimately rests; to explain the meaning of imaginary symbols, by tracing their derivation; to establish by separate and independent proofs, rules for the combination of impossible quantities, and not by inference from their similarity to rules for like combinations of real quantities; and carefully to distinguish between what is proved on evident principles, and what is only consequent from arbitrary assumptions.

Mathematical science has been at times embarrassed with contradictions and paradoxes; yet they are not to be imputed to imaginary symbols, rather than to any other symbols invented for the purpose of rendering demonstration compendious and expeditious. It may, however, be justly remarked, that

me unimportant. A useful consequence from this possibility of resolution, is said to be the integration of the form $\left(\frac{P}{Q}\dot{x}\right)$, P and Q being rational functions of x; now, when $\frac{P}{Q}$ is expressed by a series of fractions, a form as $\frac{(Ax+B)\dot{x}}{X^2+2\alpha x\quad\alpha^2+\beta^2}$ presents itself to be integrated; but the actual value of the integration cannot be assigned, without knowing what A, B are; and A, B, cannot be determined, except α, β, are known.

mathematicians, neglecting to exercise mental superintendance, are too prone to trust to mechanical dexterity; and that some, instead of establishing the truth of conclusions on antecedent reasons, have endeavoured to prop it by imperfect analogies or mere algebraic forms. On the other hand, there are mathematicians, whose zeal for just reasoning has been alarmed at a verbal absurdity; and, from a name improperly applied, or a definition incautiously given, have been hurried to the precipitate conclusion, that operations with symbols of which the mind can form no idea, must necessarily be doubtful and unintelligible. *

I have endeavoured to establish a logic for impossible quantities; to fix the meaning of certain ambiguous expressions; and to reconcile the contradictions in the doctrine of logarithms. I indulge the hope that what I have said may deter mathematicians from attempting to found demonstration on so frail and narrow a basis as analogy; or from reposing in the dangerous notion, that there are either unaccountable paradoxes, or inexplicable mysteries, in a system of characters entirely of their own invention.

* It is to be desired, that the charges of paradox and mystery, said to be introduced into algebra by negative and impossible quantities, should be proposed distinctly, in a precise form, fit to be apprehended and made the subject of discussion.